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The ruled surfaces in stone architecture

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Abstract

The stereotomic architecture is composed of ashlars made off-site and laid dry. The choice of the masonry depends on the geometrical, structural, aesthetic aspects and the mechanical properties of the material used. The fragility of the stone prevents the production of thin and sharp angles, which might break. Therefore, the angles formed by planes tangent to adjacent surfaces should tend, if possible, to a right angle. The search for the perpendicularity between the surfaces of the ashlars has brought, in stereotomy, to the widespread use of ruled surfaces (better if developable), used as a facing surfaces, but even more as junction surfaces. The choice of such surfaces is due to the fact that they could be reproduced accurately in the workshop of the stone-cutter through the movement of an auction. Although the story of the stereotomy evidences a recurring use of ruled surfaces, it is with the school of Monge that studies on these surfaces start taking shape. Among the applications of the students of Mezière some cases, brilliantly solved thanks to the properties of these surfaces, stand out.
Re-examining now part of this repertoire and transposing it into a digital environment has a dual purpose: to study and represent through the methods of mathematical representation some properties inquired to date only from the point of view of mathematical analysis; to propose, with the digital tools of design and material processing, building systems that are still highly topical.

Keywords: stereotomy, stone-cutting, descriptive geometry, ruled surfaces, developable surfaces.

1. Introduction

Stereotomy is the science that studies the cutting of solids, and has as object the construction in wood or stone cutting. Stereotomy and descriptive geometry are two deeply related sciences, since the scientific foundations of the first reside in the second one. In stereotomy knowledge of geometric entities and their properties is essential to the entire design process, and it is in the composition of the wall devices that surfaces and their properties are used.
At the end of the eighteenth century, when Monge organizes the descriptive-geometrical knowledge of the time, some geometric theories were already known, and others would be elaborated from his school or in the years to follow. The road to a synthetic theory of surfaces (studied from the point of view of pure geometry) was drawn before the Monge’s theory of descriptive geometry [9] (as deduced from the pages of *Traité de stéréotomie* written by Amédée François Frézier in the first half the eighteenth century [4]), but it is with Gaspard Monge and his school that the theory of surfaces takes the completed structure and the nomenclature that we know today.
The refining of the surface’s theory has significant implications in stereotomy. The progress of knowledge in this way allowed more and more sophisticated experiments, as deduced from the numerous devices

published in those years by the students of Monge. The control of ruled skewed surfaces and developable in particular, that were normally used in stereotomy since antiquity, allowed refined embodiments of apparatus from time to time more sophisticated.

The extent of the impact of the progress of geometry in architecture legitimizes the need of a theory that underlies the project design. In recent years, research in the field of descriptive geometry seems to be devoted to the refinement of the methods of graphic representation. So it seems having lost that original vocation, which, until the last century, made this science an indispensable tool for understanding the properties of geometric figures in space. Today, the evolution of construction techniques, the use of materials with innovative features and, last but not least, the digital tools to delegate part of the workings and operations of representation, encourages the use of complex shapes in architecture. The need for strict control of such geometries, and the capacity to imagine those, is today an occasion to rediscover the heuristic value of descriptive geometry and revisiting the tools. Today the graphical methods of representation are full flanked by digital methods, which allow to describe lines and surfaces directly in the space, in a continuous manner, with high levels of accuracy. Directly represent problems of descriptive geometry in the space, which until now were solved in the plan, allows to verify properties of lines and surfaces, and sometimes, to derive new [7].

The re-reading of the classical heritage of stone's stereotomy is addressed according to these general instances of renewal. Through the illustration of some of the relationships that ties stereotomy to descriptive geometry, we want to go through a design process that makes the knowledge of the surfaces and their properties its strength. This design practice still preserves today all its modernity, both in terms of methodological rigor, but in terms of content. The revisiting of the apparatus of classical stereotomy with the contemporary computer instrumentation has two different purposes: to trace in stereotomy the fundamentals of descriptive geometry; to revisit constructive models that, in solving complex conditions, still maintain their relevance today.

2. Ruled and developable surfaces in the school of Monge

Today we define *ruled* the surfaces generated by the movement of a straight line in space. These owe their name to the remarkable feature of admitting always the possibility of supporting on them, in all its length, a row; in geometric terms, therefore, one can always find on them a family of straight lines. This feature has been, in the past, very useful in the modeling work of ashlar stone for the vaults since their first roughing was done with the help of straight incisions.

The ruled surfaces were known for some time, as the *surfaces gauches*, before Jean Nicolas Pierre Hachette gave them their present name (The term *gauche* in French has a pejorative meaning of deformed, it seemed desirable to find Hachette a name more suited to the aesthetics of these beautiful surfaces) [6]. Hachette is the one who has first studied these surfaces extensively and has given a classification still valid today. Hachette teaches that there are two different types of surfaces generated by a straight line: *the developable surfaces which have the property of being able to be developed on a plane and ruled surfaces that are not developable*.

To understand the difference between these two great families of surfaces is necessary to clarify certain definitions and geometric properties [2]. In descriptive geometry we consider a surface as the locus of a movable curve where the constant or variable form is given in each instant, the law of motion of this curve determines the shape and position of the surface: we call this curve the *generatrix* of the surface. The generatrix may, in its motion, lean on to one or more curves called *directrix* lines. A surface is defined when, for each point of it, we can assign the generatrix line, constant or variable in shape, passing through this point.

A developable surface is the locus of tangents to a skewed directrix curve, that we name *edge of regression* of the surface; two consecutive tangents correspond to two successive positions of the straight line mobile generatrix of the surface (fig. 1). The edge of regression divides the surface into two equal and symmetrical parts. The cones and cylinders, for example, are developable surfaces where the edge of regression is a point. This point is the vertex of the cone, while in the cylinder is a point at infinity, that is a direction. The conical surface is generated by a movable generatrix line subjects to pass through a fixed point, and when the generatrix line is always parallel to itself, the conical surface becomes a cylinder.

Two subsequent lines of a developable surface include a flat element of this surface. Two successive elements are separated by a straight line of the surface and the second element can rotate with the surface around this straight line until it coincides with the plane of the first element and so on. All the elements together on the same plane form what is called the *development of the surface*. It 's evident that the development of all oblique or plane curves plotted on the developable surface, and that cut the directrices lines of this surface under certain angles, are transformed on the development plane in other curves which cut the directrices lines of the developable surface on this plane under the same angles. Moreover the development preserves the measures of the lines.

The developable surfaces are the only ones having this property of being able to be developed on a plane *without cracks or overlaps*: those plan elements have an unlimited size along the direction of the straight

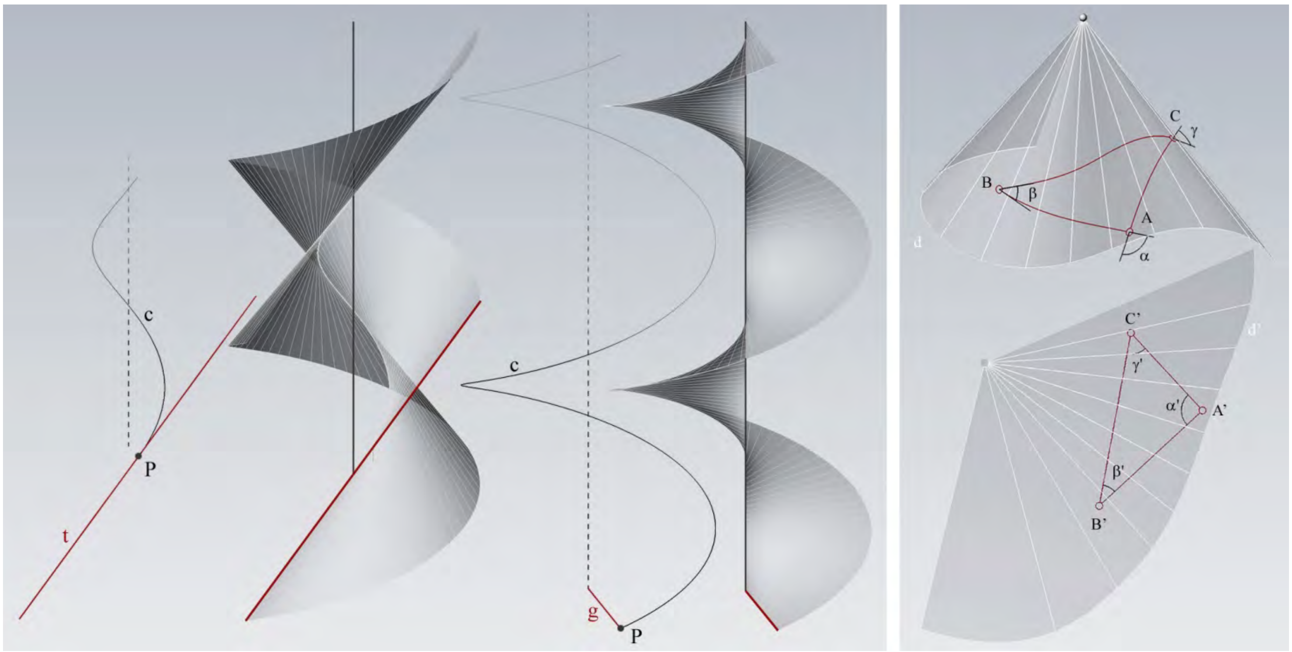

Fig. 1: From left to right; developable helical and ruled helical, genesis in comparison and properties of developable applied to the case of a generic cone.

generatrices of the surface (for the bijection, between the developed surface on the plane and developable surface in space, we need to define the surface portion to developed; in general, in fact, the surface in the development tends to overlap itself). The extensions of these elements in all directions of space, form the tangent planes of the surface. In whatever way a plan is moving in the space, the envelope of the space that runs it is a developable surface. This surface is the locus of lines, successive intersections of the moving plane.

In a ruled surface, two consecutive lines, whatever their small distance, never meet and the element between these two straight lines is not a plan; is a curved element that has an unlimited in the direction of the straight lines which include: its shape is that of an oblique plane, and for this reason have been called *skewed surfaces*. The generic ruled surface, as taught by Monge, is generated by a straight movable line (generatrix) rests on three curves dates (directrices).

The distinction between skew ruled surfaces and developable becomes even more evident in the differential classification given by Carl Friedrich Gauss (1777-1855). It is possible to classify the surfaces according to the *Gaussian curvature* (defined as the product of principal curvatures, property that belongs to the individual points of the surface). The developable ruled surfaces are those surfaces that have zero total Gaussian curvature; in other words, for differential geometry, a plane and a developable surface are part of the same class of surfaces, ie the surfaces formed by parabolic points. Instead, the skew ruled surfaces have negative Gaussian curvature and are members of another class of surfaces, ie surfaces with hyperbolic points [5].

The behavior of a tangent plane of a generic point of a ruled surface is an ultimate confirmation of the profound difference between the two types of ruled surfaces. In a skewed ruled surface the tangent plane at a point will be secant the surface along all the other points of the movable generatrix line passing through that point of contact. Instead in the developable the tangent plane of a point on the surface will be tangent to the surface along all points of the movable generatrix line passing through that point of contact. It is also said that the envelope of a developable surface can be generated by the movement of a plane in space, while the envelope of a ruled surface is given by the movement of a ruled hyperboloid in space.

For all these reasons it is natural that, in the history of descriptive geometry, the ruled surfaces formed two deeply separate chapters: the skew ruled surfaces and the developable ruled surfaces that, in summary, we define ruled and developable [1].

3. Descriptive geometric principles behind the stereotomic design

The works in cut-stone are made of ashlars out of work designed and properly laid dry on each other. The design of these prefabricated architectures are articulated around three main stages [3]:

1. the choice of the apparatus;
2. the construction of graphical or digital models of the apparatus;
3. the choice of optimal cutting methods for the realization of the ashlars.

The choice of the masonry depends on considerations of geometric-formal, structural and mechanical characteristics of the material to be used. In this regard the stereotomic design requires knowledge of the surface’s theory which make up the bodies, their property, the curves that result from their respective intersections. It also requires knowledge of methods of representation appropriate to the representation of surfaces and lines in the plane or in space. The set of this stereotomic knowledge is the science that anticipates contemporary theory and tools of descriptive geometry.

Before the advent of the computer age and therefore the spread in construction of numerical control machines, stone processing was done by hand in the stonecutter's shop. The complexity of the implementation of some surfaces influenced the choice; so the design of the surfaces, of which the ashlars were composed, was subject to their workability. Where possible were adoperate flat surfaces, alternatively, between the curved surfaces, were preferred developable, finally, the ruled.

The geometric genesis of both surfaces was simply reproducible in the workshop of the stone-cutter through the movement of a shaft in the space, to use as mould to guide the cutting operations. Unlike the ruled, however, the developable permitted the realization of *panneaux*, developments planes of the surfaces, which were made of lead or other ductile material and finally applied on the stone to guide the stonecutter in the processing. The properties of both surfaces guaranteed high accuracy in processing, with significant effects for the entire construction process. Just think of the realization of joint surfaces, that have to adhere at best with those of adjacent ashlar, required considerable rigor in the execution. Although the stereotomic construction systems are known for the dry laying of the blocks, it is frequent to encounter the use of the binders. Their sole function is to smooth the roughness of the surfaces of junction optimizing the contact surface in order to avoid phenomena of cracking caused from a bad structure.

The widespread use of ruled and developable surfaces in stone’s stereotomy is due also to the research of the perpendicularity between the joint surfaces and the facing surfaces of the ashlar. The stone is a fragile material, and this feature prevents the realization of particularly acute angles and too thin thicknesses because it would risk to break during installation even before or during processing. Furthermore if the joining surfaces of contiguous ashlars formed, with the respective facing surfaces, corners visibly disproportionate, these resist in a different manner to the stresses, and the angle more acute would tend to break compromising the stability of the structure.

The repertoire of traditional stereotomy is rich in cases in which the perpendicularity between the junction surfaces and facing surfaces is obtained by making use of ruled or developable surfaces.

4. Wall’s apparatus and ruled surfaces: the case of oblique vaults

The works that have reached us, and treatises about it, witness the widespread use of ruled and developable surfaces in stone architecture from antiquity. In particular there are different applications between the apparatus of spherical vaults, between those of the spiral staircases, among those of the splayed vaults, among those of the cylindrical oblique vaults (fig. 2) [10].

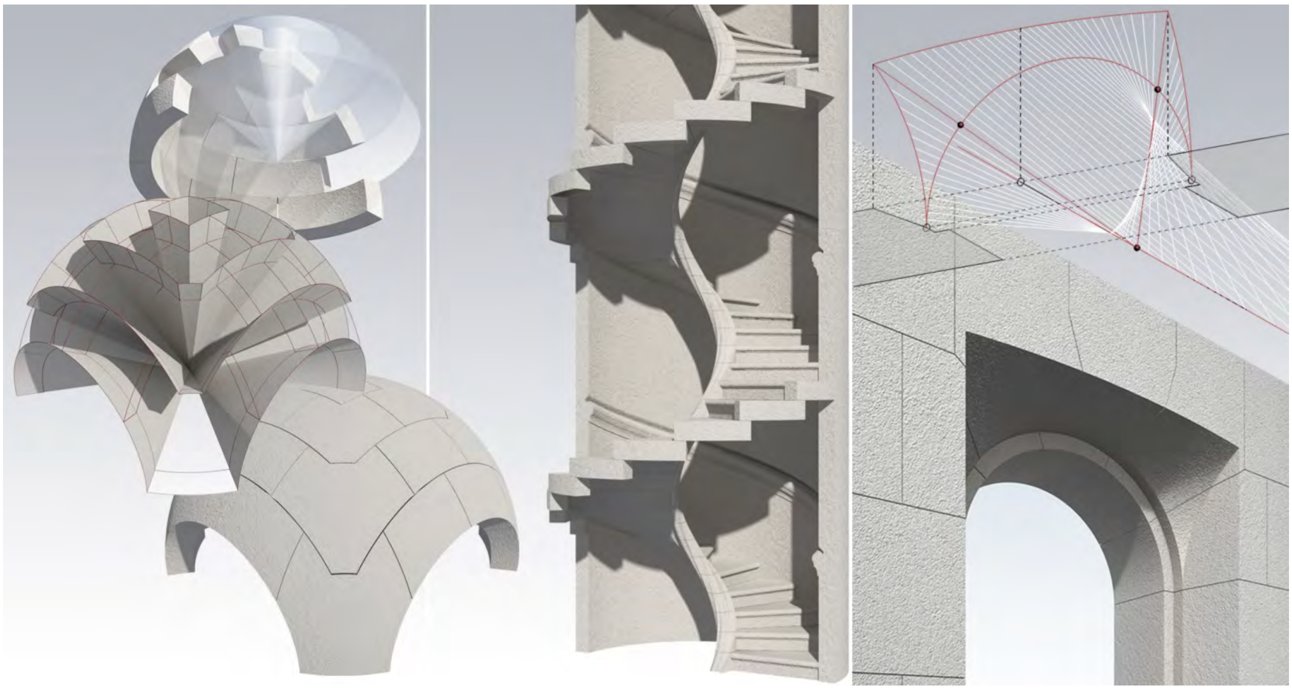

Fig.2: Developable and ruled applied to the case of spherical vaults, the spiral staircases, splayed vaults.

In the spherical vaults cases, which are among the older models, all joint surfaces are plane and developable, both in archaic apparatus and both in the more complex Renaissance. In both cases, the joining surfaces belong to quadratic cones having the center in the center of the sphere. These are maintained constantly perpendicular to the facing surface of the vault, because all normal’s surface of a sphere pass through its center. In the charming cases of spiral staircases, as the name suggests, the soffit is a ruled surface, in particular a right helicoidal, open or closed. In this kind of apparatus the step is the generating element of the form and it is a complex ashlar, consisting of surfaces of different type, flat, cylindrical, ruled. The common junction surface to two adjacent steps is a ruled helicoidal. There are also applications of ruled in a particular type of oblique vault, which takes the name of *arrière de-voussure Marseille*, generally used to cover openings of doors and windows, shaped in such a way as to ensure the opening of a door or of a frame. The surface of splayed soffit is a refined ruled surface, composed of three distinct surfaces arranged between them in continuity of tangency.

The use of ruled and developable surfaces in stereotomy finally finds a significant feedback in applications relating to a particular family of vaults, the oblique cylindrical vaults, which demonstrate the efficiency of the properties of these surfaces applied to the stone’s architecture. The cylindrical oblique vaults lie between the more complex cases of stone’s stereotomy. The interest derived in particular from structural and formal-geometrical problems that occur in conditions of pronounced obliquity. Usually, in straight vaults, the joining surfaces of the ashlars are flat and belong to the curvature lines of the cylinder intrados of the vault, ie to its generatrices and its straight sections. If you fit the same apparatus to the oblique vault case, the head ashlars will suffer from oblique forces not countered and the stability of the structure will be compromised.

In addition to the limitations of structural nature, in the design of the oblique vaults we must take account of a series of constraints related to flat sections of the hollow cylinder, which are the two elliptical frontal arches of the vault; the equal distribution of the two front arches is a geometrical-aesthetic problem difficult to solve.

These few considerations suggest the kind of problems that concern the complex geometric nature of the oblique vault case study. The history of stereotomy returns many devices that attempt to generalize and solve the issue. Some devices approximate the solution, others are particularly complex because they are composed of ashlars all different from each other that make it very difficult to process and assembly the work. Around the middle of the nineteenth century, spreads a model that is called "helicoidal apparatus". This device, due to the properties of developable and ruled, solves the problem of the oblique vault in all its generality, regardless of the geometrical shape of the vault, with geometric rigor and respect for the stereotomic principle of ashlar’s seriality.

5. Hhelicoidal apparatus

As already explained, the helical device is normally used in the case of the oblique vault: cylindrical vaults in which the axis is inclined with respect to the frontal planes, which generally are elliptical arches.

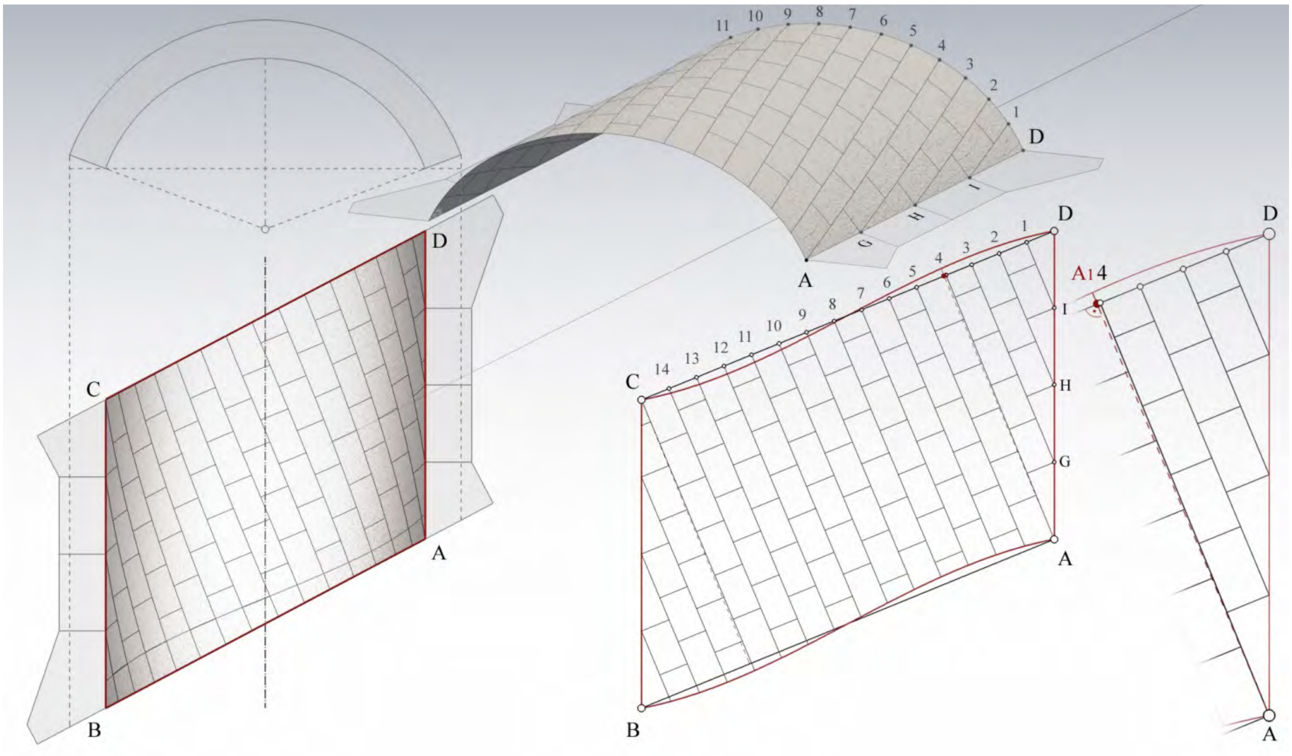

Fig.3: Development of the cylindrical surface of the intrados and construction of the junction edges of the device.

The design principle from which derives the helical device provides that the surfaces of the ashlars are all perpendicular to each other (unless very small approximations) and that the ashlars that make up the vault are all the same, condition that offers considerable savings in time during processing and assembly of the work. The apparatus which takes shape from these conditions of constraint is formed by ashlars all equal (less than the frontal ashlars and those of the impost) consisting of two cylindrical surfaces (developable), one of the intrados and one extrados, and by four helical ruled surfaces of junction, in particular from four straight helicoids (with director plan). The helices generating these helicoids, edge junction continuous and discontinuous, have the same axis of the cylindrical intrados of the vault.

The following construction, reworked in computer environment, is taken from the lessons of descriptive geometry held in 1931 by Gino Fano at the Politecnico di Torino, culminating in an essay devoted entirely to stereotomy [3].

To control this kind of device, is of great use to build the development plane of the cylindrical surface of the soffit’s vault and, in the plane, to draw the desired texture. The development plane of a oblique cylindrical surface returns a quadrilateral composed of two straight parallel lines (**AD** and **BC**) and two sinusoids (**AB** e **CD**), developments of the elliptical profiles of the arc’s face of the cylinder.

The continuous and discontinuous junction edges of ashlars are cylindrical helices, since the cylindrical helix is a geodesic curve in its development becomes a straight line (please note that the geodesic is the shortest distance between points of a surface).

The mathematical modelers allow you to automatically construct the development plan of a developable surface. Then, once you get the **ABCD** mixtilinear quadrilateral, you must determine the number of the ashlars you desire on the front of the vault: we can divide the **AB** chord into an odd number of parts, for example **15** (fig. 3).

The joining continuous edges, in the development, are the lines passing through these partition points and perpendicular to the **AB** chord. The ashlars must allocate the surface in an exact number, so the edge belonging to the junction point **A** (and point **C**) must necessarily pass through one of the partition points of the straight line **CD** (**AB** as regards **C**).

You draw the **A-A1** perpendicular to the segment **CD** and choose a point, the **4** point in figure, among the partition’s points of **CD** which is as close to **A1**. The straight line **A-4** gives the direction in the development of the continuous junction edges of the intrados.

It is a slight approximation that allows to keep the junction continues helices as much as possible perpendicular to the arches of the forehead. The edges of discontinuous junction are parallel to the chord **CD** and their arrangement is determined, ensuring the alternation, by the passage through the points **G**, **H**, **I**, intersection of the edges of continues junction with the impost edges **AD** and **BC**.

Once obtained the design of the intrados is sufficient to fold it on the surface so as to have the edges of junction in space (fig.4).

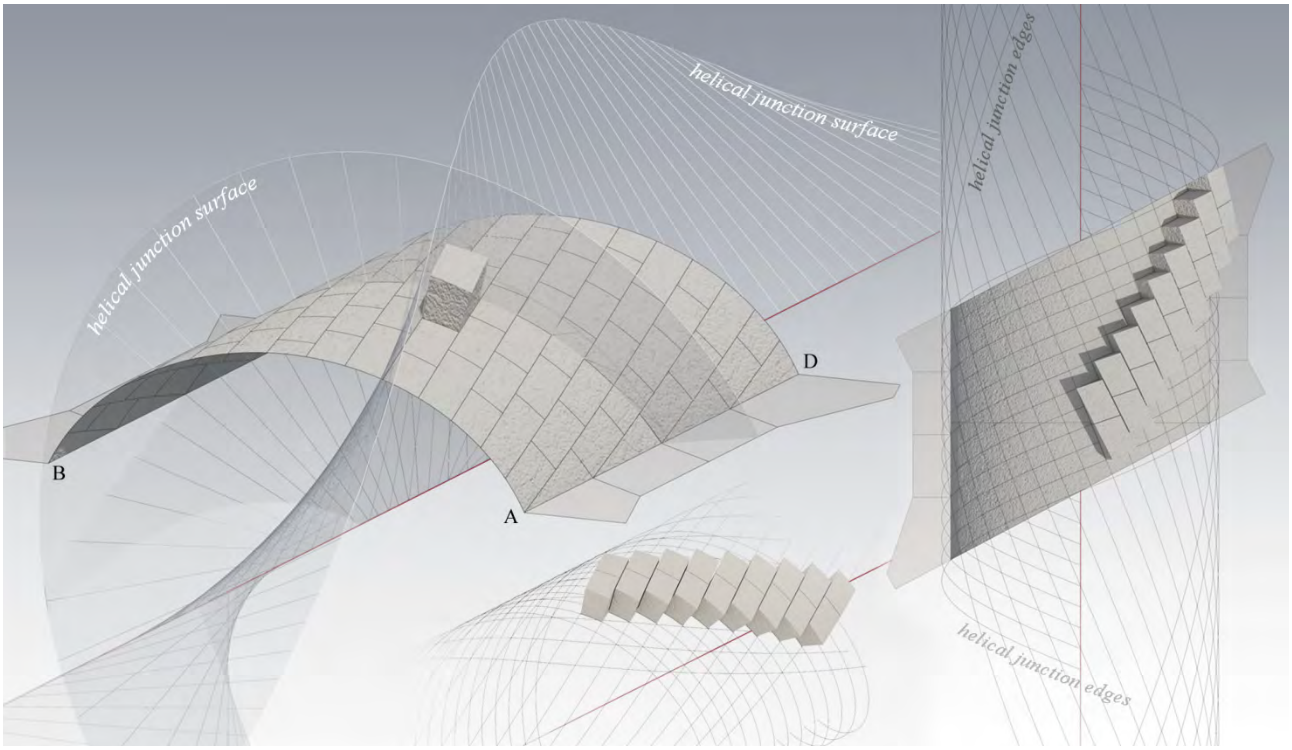

Fig. 4: The helical's serial texture is achieved through a rototranslatory movement of an ashlar type.

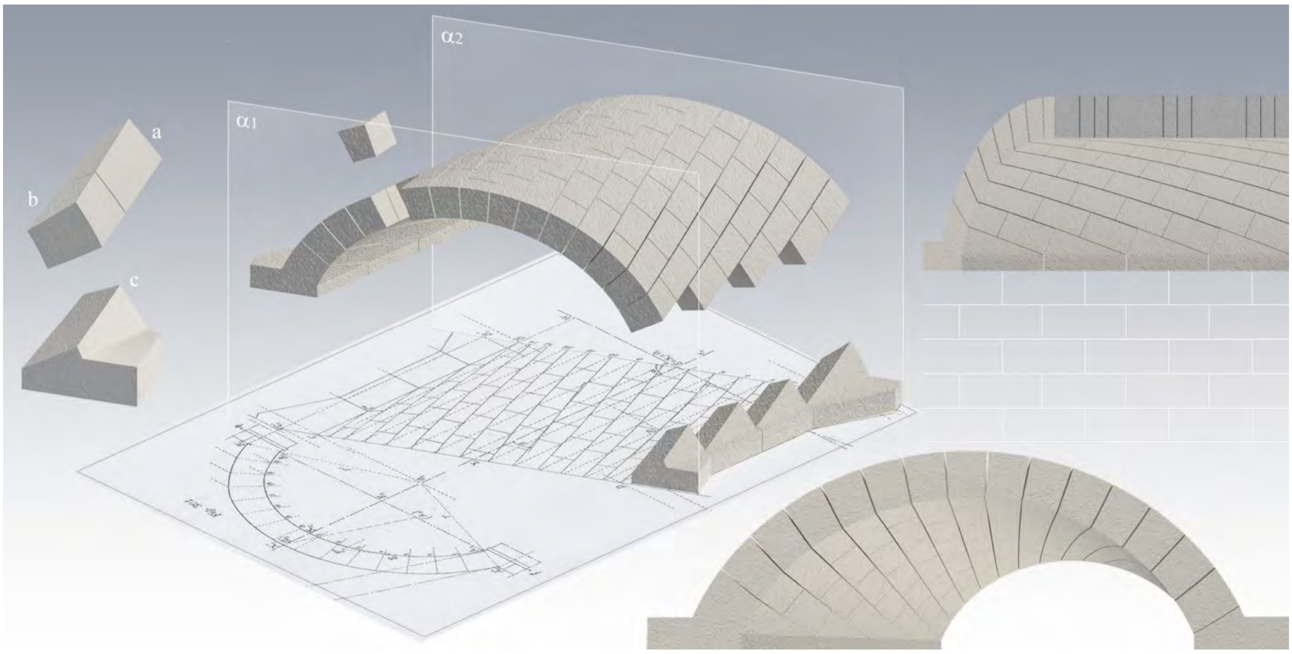

Fig. 5: Construction of the three types of ashlars that make up the apparatus.

The helical surfaces belonging to these edges and orthogonal to the cylinder of intrados dissect the extrados according to other helices. So edges and surfaces of all joints are all determined To determine the straight helicoid passing through an assigned helix, it is convenient to take, as a director of the helix and as generatrices, the two straight lines perpendicular to the surface of the intrados at the two ends of the helix.

In helical vault there are three types of different ashlars: the *interior ashlars* (type **a** in figure 5) which are the internal ones at the vault and they are all equal to each other; the *front* or *head ashlars* (type **b**), similar to those internal, but abnormal because cut along the **α** planes of the vault; the *impost ashlars* (type **c**) which rest on the piers, consisting of a horizontal base that allows a coherent insertion of the vault in the masonry and that allows to avoid the acute angle into the base. The interior impost ashlar is formed by a right prism with pentagonal section and by a further piece consists of two cylindrical triangular faces, which are those of intrados and extrados of the vault, and two straight helical surfaces (all over the ashlar is constituted by ten faces). The head ashlars are different depending on whether they are in the acute or obtuse corner of the vault over the pier.

In the acute angles the final ashlar of the vault is joined to the impost ashlar, while in the obtuse angle this can be independent. Finally it should be noted that it is always preferable that the impost ashlars have the outer surfaces perpendicular to the planes of the facade in such a way as to facilitate the connection of the vault with the wall apparatus.

6. Stereotomy principles and information technologies

The buildings of cut stone are almost completely disappeared from the shipyards of architecture for at least one hundred years (it is only used in the decorations). The abandonment of such buildings is mainly due to the introduction of reinforced concrete in construction, cheaper and more easily processable than stone. Stereotomical systems design, although no longer in use for years, retain, however, their modernity, both in solution as in the methodological rigor. Rethinking stereotomy in a contemporary way means to modernize the geometric-constructive system of ancient stereotomy through contemporary technologies. Both the technologies for the geometrical control of the shapes, as those for the automatic process of materials, contribute to the economy management of the design and construction process, and promote interesting opportunities for experimentation. The mathematical modeling, capable of describing directly in the space complex shapes in a continuous way, enables the construction of accurate digital models directly readable by the numerical control machines for cutting or, more generally, for the processing of the material. The accuracy in processing complex surfaces, characteristic of CNC machines, allows experimentation in new forms of geometry; in addition the different characteristics of this type of machines allow achievements in alternative materials to stone. The domain of experimentation and the verification is completed with the possibility of realizing rapid and low cost physical models. These prototypes, generally made in resin (can also be made of stone, wood or other material) are useful to verify the morphological and mechanical behavior of the apparatus and its parts.

With these objectives has been made the 3D printing of the helical apparatus described in the previous paragraph (the stereolithographic 3D ABS model has been printed with the Dimension Elite printer at the

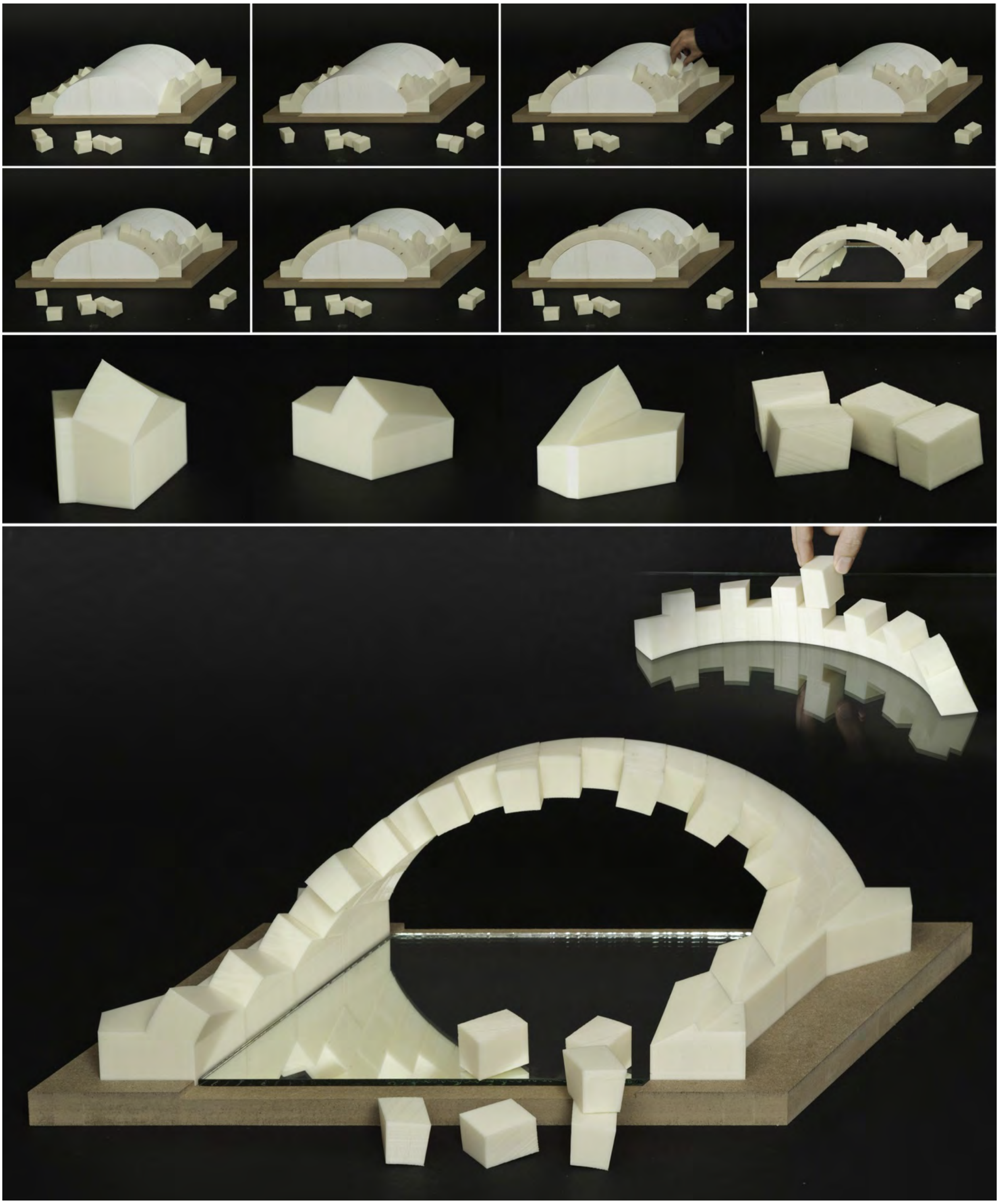

Fig. 6: Print stereolithographic 3D of the helical apparatus and relative phases of assembly.

computer lab of the Faculty of Architecture "Aldo Rossi" in Cesena, University of Bologna). In 1:50 scale, the prototype allows to verify through a physical simulation the mechanical characteristics of the vault and also verify that the individual ashlars are coherent with the apparatus in its entirety (figg.6-7). Although technological advances simplify and speed up some processes, their implementation cannot occur without a theory. The experimentation of complex forms and their realizability requires the knowledge’s theory of surfaces, their properties and methods of representation capable of representing complex shapes in space. All of this knowledge is necessary to process the apparatus from time to time more complex, designed combining modern technology and geometric-constructive principles of ancient stereotomy.

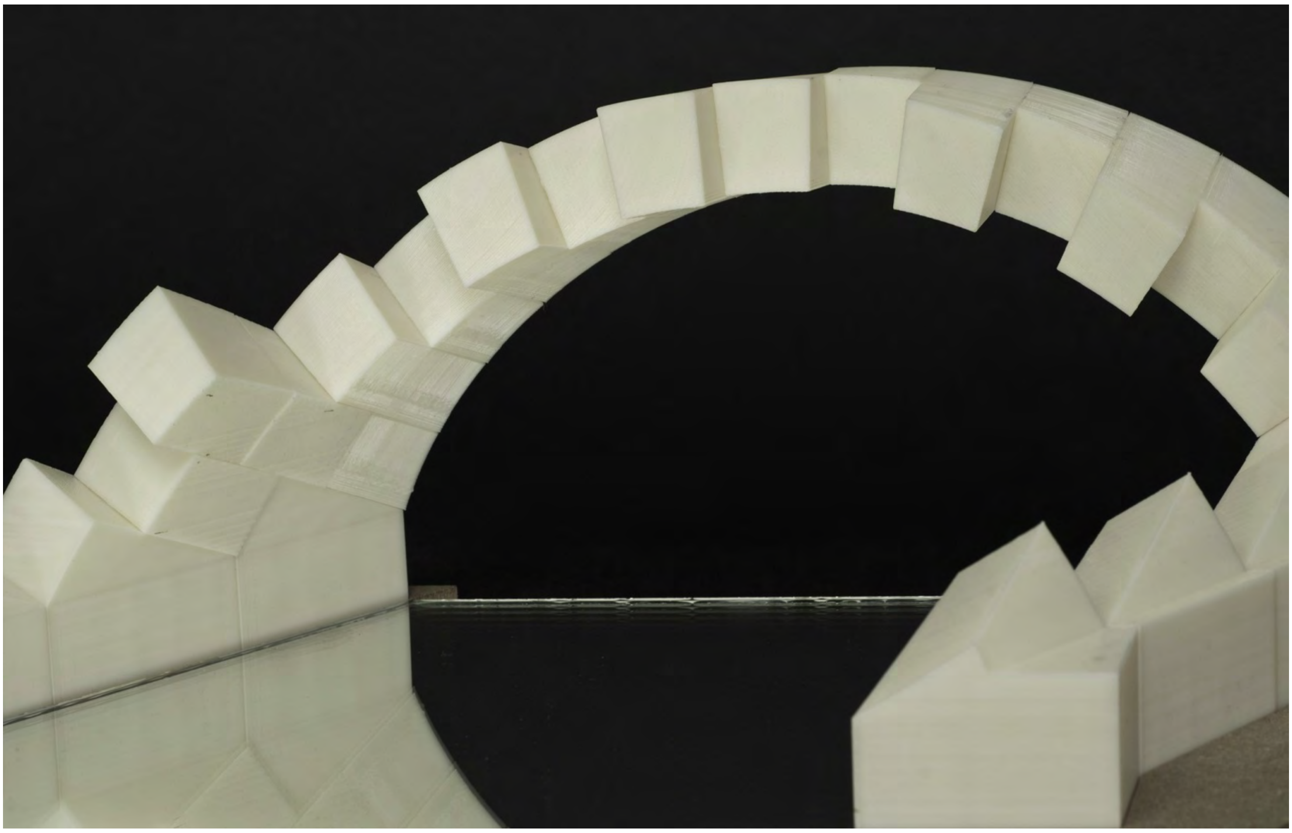

Fig. 7: Print stereolithography 3D of the helical apparatus.

Acknowledgments

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